

The plasma interaction also affects the SAHA equation through the limitation of the single particle partition function. For temperatures low enough, so that the higher terms in the partition function do not contribute this effect is of course negligible. As it becomes important with increasing temperatures the limitation is determined by  $e^2/r_0$  and not by  $e^2/D$  as suggested in Ref. <sup>9</sup>.

The above formalism requires extension where the neutral-neutral interactions are important. This is the case for the interpretation of the surprising results found from experiments under extreme pressures <sup>14</sup>.

<sup>14</sup> Y. N. RYABININ, *Gases at High Densities and Temperatures*, Pergamon Press, London 1961.

## Uniform Description of Core and Sheath of a Plasma Column

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Weakly ionized plasmas bounded by insulated walls have been studied for the limiting cases of very low and very high pressure in numerous papers. These investigations base the particle kinetics either on the model of "collision dominated motion" or on the model of "inertia dominated motion". Corresponding model regions are matched neglecting the transition region.

We consider a weakly ionized plasma bounded by insulated walls. Its particle production is due to electron collisions and its particle destruction due to wall recombination. Describing the electrons as in quasiequilibrium and ion-neutral interaction by constant mean free time collisions it is possible to derive for arbitrary plasma pressure an integrodifferential equation valid throughout the plasma core, the transition region and the sheath. No matching of model regions is required. The well-known LANGMUIR and SCHOTTKY theories are shown to be asymptotic solutions of this general treatment.

### Aim of the Investigation

Until now a uniform description of core and sheath of a plasma column exists only for the case of very low pressure, where the mean free path of the ions is much larger than the radius of the column. Since under these circumstances the ions fall freely from their point of production towards the wall the model of "inertia limited motion" is applicable throughout the whole column <sup>1, 2</sup>.

However, with increasing pressure the ions suffer collisions within the plasma column, which affect the applicability of the inertia limited model. To overcome this difficulty it is common practice to subdivide the whole column into two model regions, the "core" and the "sheath". These two model regions are matched at their interface. The core is described by the limiting kinetic model of "collision-dominated motion" <sup>3</sup>. Within the sheath the inertia

limited model of the low pressure case is applied.

It is true that for sufficiently high pressure the collision dominated approximation holds in the center of the column and the inertia limited approximation close to the walls. It is also true that both kinetic models break down in the transition region between core and sheath. The errors introduced by the core-sheath-matching which neglects the transition region are unknown.

In this investigations we aim to give for arbitrary pressures a uniform wall to wall treatment of the plasma column without invoking and matching model regions. This is done for a special case which allows the desired accuracy and is still close to reality. The theories of LANGMUIR and SCHOTTKY valid in the low and high pressure limits respectively are shown to be asymptotic solutions of this general theory.

<sup>1</sup> J. LANGMUIR and L. TONKS, *Phys. Rev.* **34**, 876 [1929].

<sup>2</sup> S. A. SELF, *Phys. Fluids* **6**, 1762 [1963].

<sup>3</sup> W. SCHOTTKY, *Phys. Z.* **25**, 635 [1924].



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### Model

We consider a one-dimensional stationary plasma consisting of one kind of neutrals (0), ions of the same kind (+) and electrons (-). The plasma is bounded by two planar walls perpendicular to the  $x$ -direction with a distance  $2L$ . The external electric field producing the discharge is parallel to the walls and constant.

We assume that the degree of ionization is so small that the neutrals are close to equilibrium and consequently can be described by constant density  $n_0$  and a MAXWELLIAN distribution

$$F_0(w) = \left( \frac{M}{2\pi k T_0} \right)^{1/2} \cdot \exp \left( -\frac{M w^2}{2 k T_0} \right) \quad (1)$$

( $w$  =  $x$ -component of the velocity  $v$ ;  $M$  = atomic mass;  $k$  = BOLTZMANN's constant;  $T_0$  = temperature of the neutrals).

Under these conditions the electron temperature  $T_-$  can be treated as constant but may be different from  $T_0$ .

Volume production is due only to electron collisions with neutrals and therefore proportional to the electron density  $n_-$ . Volume recombination is negligible. The particles recombine at the insulated walls.

The ions move freely under the influence of the electric field suffering collisions with neutrals only. These collisions are assumed to be practically all charge exchange collisions with constant mean free time. We do *not* invoke the concept of mobility or diffusion. The electrons are assumed to be in quasi-equilibrium.

The conditions of this model are not far from the reality of many experimental situations.

For insulated walls the assumption of quasi-equilibrium is a reasonable concept throughout the whole region of interest except for very high energies close to the wall. This deviation affects the electron number density  $n_-$  only negligibly. However, it is important for the production coefficient  $\alpha$ , which is governed by the high energy electrons. Fortunately this uncertainty of  $\alpha$  has no consequences for our investigation since only  $\alpha n_-$  enters, which is very small near the wall. The assumption of charge exchange collisions with constant mean free time for the ions is open to question. The ions move in their own gas and charge exchange is therefore indeed the dominating collision process. Moreover ions and neutrals interact on large distances

according to the law of MAXWELLIAN molecules, for which the "mean free time concept" is correct. So our collision concept is probably as close to the true behaviour of a real system as one can get without introducing much more detailed collisional concepts.

But even if experimental realizability of our model were not too good, the proposed investigation would still be valuable in that it allows the first comparison between a rigorous treatment and the usual "multiple-model-zone approximation".

### Concept

The key advantage of our model lies in the fact that the velocity distribution of the ions after production or charge exchange collisions is identified with that of the neutrals, which is known to be MAXWELLIAN.

This allows us to derive an integral equation for the ion distribution function  $f$ , which depends on the potential distribution  $U$  and the electron density  $n_-$ . It is found by classing all ions in a given test volume element ( $x$ ) according to their velocity  $w$  and the location  $\xi$  of their last collision. The number of ions emerging from a collision at  $\xi$  with the velocity  $\omega$  can be expressed with the help of the electron and ion densities at  $\xi$ . The velocity  $\omega$  with which they emerge from their last collision is related to the velocity  $w$  at the test point  $x$  through the law of inertia limited motion in the electric field. Integrating over all velocities we find the integral equation for the ion density.

Due to the electron-quasi-equilibrium the electron density is related to the potential by BOLTZMANN's formula.

POISSON's equation provides another connection between  $f$ ,  $n_-$  and  $U$ .

These three relations allow to solve the problem.

### The Ion Distribution Function

We consider the ion density at  $x$  (see Fig. 1). The number of ions emerging from a charge exchange collision in the volume element  $S d\xi$  at  $\xi$  with a velocity component ( $\omega, d\omega$ ) in the  $x$ -direction is

$$\Delta_c = n_0 F_0(\omega) d\omega n_- \overline{Q_c v} S d\xi. \quad (2)$$

$S$  denotes the area of the walls which we consider in the limiting case  $S \rightarrow \infty$ .  $Q_c$  designates the cross

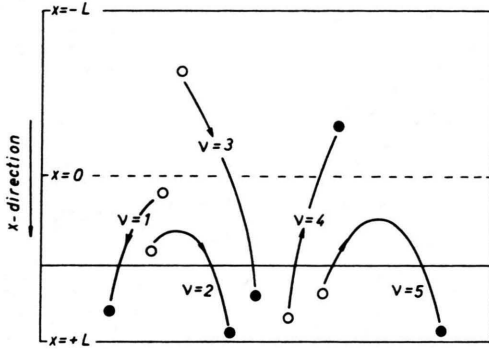


Fig. 1. Schematic ion paths. (○) denotes last collision characterized by  $(\xi, \omega, \eta)$ . (●) denotes next collision after crossing the test volume, which is characterized by  $(x, w, U)$ .

section for charge exchange collisions and the bar in  $\overline{Q_c v}$  indicates the average over the ion velocity distribution. Since we are dealing with "constant mean free time collisions" the product  $Q_c v$  is independent of the velocity and therefore we do not need the velocity distribution of the ions at  $x$  to calculate  $\overline{Q_c v}$ . This decisive advantage yields

$$\overline{Q_c v} = Q_c v = \text{const} = \nu/n_0; \quad (3)$$

$\nu$  being the constant collision frequency of the single ion.

The number of ions produced by electron-neutral impact in the volume element  $S d\xi$  at  $\xi$  with a velocity component  $(\omega, d\omega)$  is

$$\Delta_i = n_0 F_0(\omega) d\omega n_- \overline{Q_i v_-} S d\xi, \quad (4)$$

where  $Q_i$  is the ionization cross section of the electrons and the bar in  $\overline{Q_i v_-}$  indicates the average over the electron velocity distribution, which is MAXWELLIAN. Eq. (4) implies, that the ionization process does not severely change the velocity distribution of the neutrals, which is justified except close to the wall where the argument of page 2028 holds.

$$n_0 \overline{Q_i v_-} = \alpha \quad (5)$$

we call the ionization coefficient.

The fraction of the ions  $\Delta = \Delta_c + \Delta_i$  which passes through  $x$  without having suffered a new collision is

$$\Delta \exp\{-\nu t\}. \quad (6)$$

$t$  being the time of flight from  $\xi$  to  $x$  which of course depends on the initial velocity  $\omega$  and on the potential distribution  $U$  through the relation

$$t = \int \frac{du}{du/dt} = \int \frac{du}{(e/M) E(u)} \quad (7)$$

with the electric field  $E = -dU/dx$  considered as a function of the velocity component  $u$  during the free flight, which is related to the potential by

$$\frac{M}{2} u^2 + e U(x_u) = \frac{M}{2} w^2 + e U(x) = \frac{M}{2} \omega^2 + e U(\xi). \quad (8)$$

Thus the number of ions at  $x$  with the velocity  $v$  which had their last collision at  $\xi$  is

$$\begin{aligned} & \frac{\Delta \exp\{-\nu t\}}{|w|} \\ &= S \frac{d\omega d\xi F_0(\omega)}{|w|} (\nu n_+(\xi) + \alpha n_-(\xi)) \exp\{-\nu t\}. \end{aligned} \quad (9)$$

Integrating over all contributions  $\xi$  we find for the ion distribution function  $f$

$$\begin{aligned} f(x, w) dx dw = \sum_i dx dw D_i(w) \int_{-L}^{+L} d\xi D_i(\xi) \frac{F_0(\omega)}{|w|} \\ \cdot (\nu n_+(\xi) + \alpha n_-(\xi)) \cdot \exp\left\{-\nu \int_{-\infty}^{+\infty} \frac{du D_i(u)}{(e/M) E(u)}\right\} \end{aligned} \quad (10)$$

where the  $D_i$ -functions are DIRICHLET-like functions defined in Eq. (11).

The right hand side of (10) is composed of contributions of identical structure. Each of them belongs to another type of ion path the qualities of which are reflected in the limits of  $\xi, w$  and  $u$  enforced by the functions  $D_i$ . For an arbitrary potential distribution there might be many such terms. Fortunately in our case the potential distribution is simple in that it has a maximum in the center at  $x=0$  and decreases monotonically towards the wall. In this case we have five contributions. The corresponding ion paths are shown schematically in Fig. 1.

The functions  $D_i$  are zero everywhere except in the following intervals, where they have the value one

$\begin{aligned} i \leq 1: & \quad 0 \leq \xi \leq x \\ i \leq 2: & \quad 0 \leq \xi \leq x \\ i \leq 3: & \quad -L \leq \xi \leq 0 \\ i \leq 4: & \quad x \leq \xi \leq L \\ i \leq 5: & \quad x \leq \xi \leq L \end{aligned}$	$\begin{aligned} \sqrt{2(e/M)(\eta-U)} \leq w \leq \infty \\ \sqrt{2(e/M)(\eta-U)} \leq w \leq \sqrt{-2(e/M)U} \\ \sqrt{-2(e/M)U} \leq w \leq \infty \\ -\infty \leq w \leq 0 \\ 0 \leq w \leq \sqrt{-2(e/M)U} \end{aligned}$	$\left\{ \begin{aligned} \omega \leq u \leq w \\ -\omega \leq u \leq w \\ -\sqrt{\omega^2 + 2(e/M)\eta} \leq u \leq w \\ \sqrt{\omega^2 + 2(e/M)\eta} \leq u \leq w \\ -\infty \leq u \leq w \\ -\infty \leq u \leq w \end{aligned} \right. \quad (11)$
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Here we used the abbreviations

$$U = U(x) \quad \text{and} \quad \eta = U(\xi).$$

From (10) we obtain for the ion number density  $n_+$

$$n_+(x) = \int_{-\infty}^{+\infty} dw f(x, w) \quad (12)$$

and for the ion current density  $j_+$  in the  $x$ -direction

$$j_+(x) = \int_{-\infty}^{+\infty} dw w f(x, w). \quad (13)$$

Since there is no particle destruction within the volume the particle conservation law yields

$$j_+(x) = \int_0^x d\xi \alpha n_-(\xi). \quad (14)$$

### Electron Density

The quasi-equilibrium of the electrons provides the simple connection

$$n_- = n_c e^{+eU/kT_-} \quad (15)$$

$n_c$  is the electron density at the center  $x=0$  where  $U=0$ .

### Poisson Equation

According to our concept the third equation used in the solution of our problem is Poisson's equation

$$n_+ = n_- - \frac{\epsilon_0}{e} \frac{d^2 U}{dx^2}. \quad (16)$$

### The Uniform "Core-Sheath-Equation"

Eqs. (13) and (14) yield with (10)

$$\sum_i \int_{-L}^{+L} d\xi D_i(\xi) \int_{-\infty}^{+\infty} d\omega D_i(\omega) \frac{\omega}{|w|} (\nu n_+(\xi) + \alpha n_-(\xi)) F_0(\omega) \cdot \exp \left\{ + \nu \int_{-\infty}^{+\infty} \frac{du D_i(u)}{(e/M) dU/dx|_u} \right\} = \int_0^x d\xi \alpha n_-(\xi). \quad (17)$$

Together with (15) and (16) this is the *general "core-sheath-equation"*, which is an integral equation for  $U(x)$ .

In deriving (17) we have changed from the variable  $w$  to  $\omega$  using

$$w dw = \omega d\omega \quad (18)$$

which follows from Eq. (8). Due to this transition one has to replace  $D_i(w)$  by  $D_i(\omega)$ , which is not vanishing in the intervals

$$\begin{aligned} i \leq 1: & \quad 0 \leq \omega \leq \infty, \\ i \leq 2: & \quad -\sqrt{-2(e/M)\eta} \leq \omega \leq 0, \\ i \leq 3: & \quad \sqrt{-2(e/M)\eta} \leq \omega \leq \infty, \\ i \leq 4: & \quad -\infty \leq \omega \leq -\sqrt{2(e/M)(\eta-U)}, \\ i \leq 5: & \quad -\sqrt{-2(e/M)\eta} \leq \omega \leq -\sqrt{2(e/M)(\eta-U)}. \end{aligned} \quad (19)$$

### Boundary Conditions

If we choose the potential  $U=0$  in the center and remember the requirement of regularity we have the condition

$$x=0: \quad U=0, \quad E = -dU/dx = 0. \quad (20)$$

At the insulated wall we must secure zero electric current

$$x = \pm L: \quad j_+(\pm L) - \frac{n_c \bar{v}_-}{4} \exp \left\{ \frac{e U(\pm L)}{k T_-} \right\} = 0. \quad (21)$$

### The Cold Gas Approximation

It is quite obvious that the general core-sheath equation given in Eq. (17) presents an extremely difficult mathematical problem.

A remarkable simplification is obtained in the limiting case of a cold gas,  $T_0 \rightarrow 0$ . As is readily shown from the general equation (17) this approximation neglects terms of the order of  $T_0/T_-$  and smaller in comparison to 1. Since the gas temperature in general is much lower than the electron temperature the "Cold Gas Approximation" is an approximation of great practical interest.

The "core-sheath-equation" reads then

$$j_+(x) = \int_0^x d\xi \alpha n_-(\xi) \quad (22)$$

$$= \int_0^x d\xi (\alpha n_-(\xi) + \nu n_+(\xi)) \exp \left( -\nu \int_0^w \frac{du}{(e/M) E(u)} \right)$$

with  $w = \sqrt{2(e/M)(U(\xi) - U(x))}$ .

In comparison to Eq. (17) the much simpler form of Eq. (22) is due to the fact that it was possible to carry out the integration over  $\omega$  and that we have only one type of ion path left ( $i=1$ ).

### Langmuir and Schottky Theories as Asymptotic Solutions

Remembering the assumptions underlying LANGMUIR's<sup>1</sup> *inertia limited theory* we should expect that this result follows from the "cold gas approximation" for the asymptotic case  $\nu \rightarrow 0$ . If we develop Eq. (22) in a power series with respect to  $\nu$  we obtain in the lowest order

$$\int_0^x d\xi \left( n_+(\xi) - \alpha n_-(\xi) \int_0^w \frac{du}{(e/M) E(u)} \right) = 0 \quad (23)$$

which after differentiation with respect to  $x$  results in

$$n_+(x) = \int_0^x \frac{\alpha n_-(\xi) d\xi}{w}. \quad (24)$$

This is the well-known LANGMUIR relation.

The set of Eqs. (15), (16), and (24) is open to a machine solution as has been demonstrated by SELF<sup>2</sup>.

For the quasineutral case ( $n_+ - n_- \ll n_+ + n_-$ ) a closed analytical solution beyond the expansions of LANGMUIR is possible.

Introducing Eq. (15) into (24) and considering  $x$  as a function of  $y = -eU/kT_-$  we obtain

$$\int_0^y d\bar{y} \frac{dx}{dy} \Big|_{\bar{y}} \cdot \frac{\exp\{y-\bar{y}\}}{\sqrt{y-\bar{y}}} = \frac{1}{\alpha} \sqrt{2 \frac{kT}{M}} = A. \quad (25)$$

We solve this integral equation by LAPLACE transform<sup>4</sup>, which yields for Eq. (25)

$$XY \sqrt{\frac{\pi}{y-1}} = \frac{A}{Y} \quad (26)$$

where  $X(Y)$  is the LAPLACE transform of  $x(y)$ . Eq. (26) can be solved for  $X$ . Backtransformation of  $X(Y)$  yields

$$\begin{aligned} x(y) &= \frac{A}{\pi} e^y \int_0^\infty dY e^{-Y} \sqrt{Y} \sin 2\sqrt{Y} Y \\ &= \frac{2}{\pi} A \sqrt{y} \left\{ \left( \sqrt{y} - \frac{1}{2\sqrt{y}} \right) \int_0^{\sqrt{y}} dY \exp\{Y^2\} - \frac{1}{2} e^y \right\} \\ &= -\frac{2}{\pi} A \sqrt{y} \sum_{n=0}^\infty \frac{y^n}{n!(4n^2-1)}. \end{aligned} \quad (27)$$

SCHOTTKY's *collision dominated theory*<sup>3</sup> results from our general description as the asymptotic solution for  $\nu \rightarrow \infty$ .

To show this it is advised to subdivide the interval  $(0, x)$  of the  $\xi$ -integration in Eq. (22) into the two sub-intervals  $(0, x-\varepsilon)$  and  $(x-\varepsilon, x)$ . Here  $\varepsilon$  is a small quantity for which limiting conditions are given in the following.

The integration over the first interval  $(0, x-\varepsilon)$  contributes

$$j_1 = \delta x n_+(0) \exp \left\{ -\nu \frac{\sqrt{2(e/M)} (U(x-\varepsilon) - U(x))}{(e/M) E(x)} \right\} \quad (28)$$

where  $\delta$  is a positive quantity smaller than one.

The integration over the second interval  $(x-\varepsilon, x)$  produces

$$j_2 = \int_{x-\varepsilon}^x d\xi \nu n_+(x) \exp \left\{ -\nu \sqrt{\frac{2(x-\xi)}{(e/M) E(x)}} \right\} \quad (29)$$

provided  $\varepsilon$  is chosen such that the conditions:

$$\varepsilon \ll \frac{E}{dE/dx}; \quad \varepsilon \ll \frac{n_+}{dn_+/dx} \quad (30)$$

are fulfilled.

Carrying out the elementary integration in Eq. (29) and adding (28) we find

$$\begin{aligned} j_+(x) &= \frac{e}{M\nu} n_+(x) E(x) \\ &\cdot \left\{ 1 - \left( 1 + \nu \sqrt{2 \frac{M\varepsilon}{eE}} + \nu^2 2 \frac{M\varepsilon}{eE} \frac{\delta}{2} \frac{n_+(0)}{n_+(x)} \right) \right. \\ &\cdot \exp \left\{ -\nu \sqrt{2 \frac{M\varepsilon}{eE}} \right\} \left. \right\}. \end{aligned} \quad (31)$$

The additional terms in the bracket disappear if  $\varepsilon$  can be chosen such that

$$\varepsilon > \frac{1}{\nu^2} \frac{eE}{2M} \quad (32)$$

holds.

The condition (30) and (32) are adverse and consequently a suitable quantity  $\varepsilon$  does only exist if the condition

$$\left( \frac{dE}{dx} \right) \ll 2 \frac{M}{e} \nu^2 \cong \frac{E}{l_+} \quad (33)$$

is fulfilled. For the asymptotical case  $\nu \rightarrow \infty$  this requirement is always met and therefore we find the representation

$$j_+(x) = \frac{e}{M\nu} n_+(x) E(x) \quad (34)$$

for the current density\*.

\* It should not surprise that the diffusion term is not present, since due to our basic assumption  $T_0=0$  the ions have no thermal velocity.

<sup>4</sup> A. ERDÉLYI, Tables of Integral Transforms, Vol. 1, McGraw-Hill, New York 1954.



Using Eq. (34) together with (14), (15) and the concept of quasineutrality we obtain SCHOTTKY's result for a quasineutral collision dominated plasma core

$$\alpha n_- = \frac{d}{dx} \frac{e}{M \nu} n_+ E = - \frac{k T_-}{M \nu} \frac{d^2 n_+}{dx^2} \quad (35)$$

with the solution

$$n_{\pm} = n_c \cos \left( \frac{\pi}{2} \frac{x}{L} \right) \quad (36)$$

and the eigenvalue

$$\alpha = \frac{k T_-}{M \nu} \left( \frac{\pi}{2} \right)^2. \quad (37)$$

### General Solution

It was shown in the preceding paragraph that LANGMUIR's and SCHOTTKY's theories are correct only asymptotically in the limits  $\nu \rightarrow 0$  and  $\nu \rightarrow \infty$  respectively. For finite values of  $\nu$  it is to be expected that they are subject to corrections which are unknown.

Our general core-sheath equation (17) respectively (22) allows to calculate these corrections. This requires the solution of the eigenvalue problem described by the integral equation (17) respectively (22).

There is little hope to find an analytical approach. For a numerical solution with a digital computer one would first choose a specific set of experimental

data and approach the corresponding solution by an iterative procedure, starting from an appropriate initial distribution. Since we are aiming to find the deviations of our rigorous theory from the elementary multiple-model-region approach, it is advised to use as such a starting distribution the above described composition of the collision dominated solution for the plasma core and the inertia dominated solution in the sheath at the wall.

### Conclusion

We have derived a core-sheath-equation (17), which describes within the frame of our model rigorously and uniformly the whole system including core and sheath for arbitrary pressures. The theories of LANGMUIR and SCHOTTKY are shown to be asymptotic solutions of this general description.

The general equation is of a rather complicated nature. This is not surprising since it incorporates many involved problems. It is not only able to describe the transition region and the plasma column for intermediate pressures, but it also is able to describe the motion under the influence of very high or very inhomogeneous fields and density gradients.

### Acknowledgement

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